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LETTER TO THE EDITOR

Boson realization of non-generic $sl_q(2)$-R matrices for the Yang–Baxter equation

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Abstract. By establishing a new Boson realization of quantum universal enveloping algebra $sl_q(2)$ and its representations in the non-generic case that q is a root of unity, we systematically construct non-generic R-matrices of $sl_q(2)$ through the universal R-matrix. These new R-matrices are not covered by the standard R-matrices constructed in terms of quantum group and the non-standard ones obtained by using the extended Kauffman's diagram technique.

The Yang–Baxter equation plays a crucial role in nonlinear integrable systems in physics [1–3], and its solutions can be constructed in terms of the quantum universal enveloping algebra (QUEA) $U_q(L)$ of a classical Lie algebra L [4–6]. Some results and notions, which will be used in this letter, are briefly reviewed as follows.

Let $\{e_a\}$ be the basis for a certain Borel subalgebra of $U_q(L)$ and $\{e^a\}$ be its dual, then for a given representation ρ of $U_q(L)$, the Hopf algebraic structure of $U_q(L)$ ensures

$$R=\rho\otimes\rho(\mathscr{R})=\rho\otimes\rho\left(\sum_a e_a\otimes e^a\right)=\sum_a \rho(e_a)\otimes\rho(e^a)$$

to satisfy the Yang–Baxter equation without spectral parameter

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{1}$$

where $\mathscr{R}=\Sigma_a e_a\otimes e^a$ is called universal R-matrix and

$$R_{12}=\sum_a \rho(e_a)\otimes\rho(e^a)\otimes I$$

$$R_{13}=\sum_a \rho(e_a)\otimes I\otimes\rho(e^a)$$

$$R_{23}=\sum_a I\otimes\rho(e_a)\otimes\rho(e^a).$$

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If ρ is chosen to be irreducible, R is called a standard R-matrix, which can be expressed by q-C-G coefficients [7]. In the family of R-matrices, besides the standard R-matrices there are non-standard ones associated with Lie algebras A_n, B_n, C_n, D_n. They are systematically constructed by extending Kauffman's diagram technique [8], and its possible relations to QUEA or quantum group have been analysed [9].

In this letter, we will briefly report the construction of a new class of R-matrices essentially different from the above mentioned two classes of R-matrices. Further results based on this letter will be published later.

On the q-Fock space [10–12] $\mathscr{F}_q$

$$\{F(n)=a^{+n}|0\rangle | N|0\rangle = a|0\rangle = 0 \qquad n\in \mathbb{Z}^+=\{0,1,2,\ldots\}\}$$

where boson operators a^+, $a=a^-$ and N satisfy the q-deformed boson commutation relations

$$aa^+-q^{-1}a^+a=q^N \qquad [N,a^\pm]=\pm a^\pm \qquad [a^\pm,a^\pm]=0 \tag{2}$$

the operators

$$J_+=\frac{1}{[2]_q}a^{+2} \qquad J_-=-\frac{1}{[2]_q}a^2 \qquad J_3=N+\tfrac{1}{2} \tag{3}$$

satisfy

$$[J_+,J_-]=[J_3]_{q^2} \qquad [J_3,J_\pm]=\pm 2J_\pm \tag{4}$$

where $[f]_t=(t^f-t^{-f})/(t-t^{-1})$. Thus, (3) gives a new realization of the QUEA $\mathrm{sl}_q(2)$, which is inhomogeneous and an embedding into the q-deformed boson realization of $(C_n)_q$ [13].

A natural representation Γ are defined on $\mathscr{F}_q$ as

$$J_\pm F(n)=\pm[2]^{-1}([n][n-1])^{\frac{1}{2}(|\mp|)}F(n\pm 2) \qquad J_3F(n)=(n+\tfrac{1}{2})F(n). \tag{5}$$

It is easy to observe that $\mathscr{F}_q$ and Γ can be reduced as

$$\Gamma=\Gamma^+\oplus\Gamma^- \qquad \mathscr{F}_q=\mathscr{F}_q^+\oplus\mathscr{F}_q^-$$
$$\mathscr{F}_q^\pm=\{f^\pm(m)=F(2m+\tfrac{1}{2}(|\mp|))|m\in\mathbb{Z}^+\}$$

where Γ^+ and Γ^- are representations on invariant subspaces $\mathscr{F}_q^+$ and $\mathscr{F}_q^-$ respectively. From (5) we can write down the explicit forms of Γ^+ and Γ^-

$$\begin{aligned} J_+f^\pm(m)&=[2]^{-1}f^\pm(m+1)\\ J_-f^\pm(m)&=-[2]^{-1}[2m+\tfrac{1}{2}(|\mp|)][2m+\tfrac{1}{2}(|\mp|)-1]f^\pm(m-1)\\ J_3f^\pm(m)&=(2m+\tfrac{1}{2}(|\mp|)+\tfrac{1}{2})f^\pm(m).\end{aligned} \tag{6}$$

It can be proved that $\Gamma^\pm$ are irreducible when q is generic and indecomposable (reducible but not completely reducible) when q is a root of unity [14]. For the non-generic case we have $q^p=1$ where p is an integer larger than or equal to 3, so $[\alpha p]=0(\alpha\in\mathbb{Z}^+)$, therefore there exist $\Gamma^\pm$-invariant subspaces

$$V^\pm_{\alpha p}=\left\{f^\pm(m)\middle| m\geqslant\frac{\alpha p\pm\sigma(\alpha p)}{3}\right\}$$

determined by the extreme vector $f^\pm(\frac{1}{2}(\alpha p\pm\sigma(\alpha p)))$ satisfying $J_-f^\pm(\frac{1}{2}(\alpha p\pm\sigma(\alpha p)))=0$ where $\sigma(x)=\frac{1}{2}(1-(-1)^x)$.

Now one can easily see that on the quotient spaces $Q_\alpha^\pm(p)=\mathcal{F}_q^I/V_{\alpha p}^\pm$:

$$\{|j,M\rangle^\pm=\bar{f}^\pm(j+M)|j=\tfrac{1}{4}(\alpha p\pm\sigma(\alpha p)-2),\ M=j,j-3,\ldots,j\}$$

$\Gamma^\pm$ induces $\frac{1}{2}(\alpha p\pm\sigma(\alpha p))$-dimensional representations of $\mathrm{sl}_q(2)$

$$J_+|j,M\rangle^\pm=[2]^{-1}|j,M+1\rangle \qquad J_+|j,j\rangle=0$$
$$J_-|j,M\rangle^\pm=-[2]^{-1}[2(j+M)+\tfrac{1}{2}(|\mp|)][2(j+M)+\tfrac{1}{2}(|\mp|)-1]|j,M-1\rangle^\pm \tag{7}$$
$$J_3|j,M\rangle^\pm=(2(j+M)+1\mp\tfrac{1}{2})|j,M\rangle^\pm.$$

The above representations are completely new and not covered by the standard angular momentum representation with $q^p=1$, and their relations to the indecomposable representations induced by the regular representation [13] are still unknown.

The explicit matrices of the representation on $Q_2^+(3)$ are .

$$J_+=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\qquad J_-=\begin{bmatrix}0&0&0\\0&0&0\\0&-1&0\end{bmatrix}\qquad J_3=\frac{1}{2}\begin{bmatrix}9&0&0\\0&5&0\\0&0&1\end{bmatrix}. \tag{8}$$

For $p=5$ and $\alpha=1$, we have the representation

$$J_+=\begin{bmatrix}0&[2]^{-1}&0\\0&0&[2]^{-1}\\0&0&0\end{bmatrix}\qquad J_-=\begin{bmatrix}0&0&0\\-1&0&0\\0&-1&0\end{bmatrix}\qquad J_3=\frac{1}{2}\begin{bmatrix}9&0&0\\0&5&0\\0&0&1\end{bmatrix} \tag{9}$$

on the space $Q_1^+(5)$.

Through the universal $\mathrm{sl}_q(2)$ R-matrix

$$R=q^{J_3\otimes J_3}\sum_{n=0}^{\infty}\frac{(1-q^{-4})^n}{[n]q^2!}q^{n(n-1)}(q^{J_3}J_+\otimes q^{-J_3}J_-)^n \tag{10}$$

the boson realization $\mathcal{R}_B\in E_{nd}(\mathcal{F}_q\otimes\mathcal{F}_q)$ of the R-matrix is given by

$$\mathcal{R}_B=q^{(N+\frac{1}{2})\otimes(N+\frac{1}{2})}\sum_{n=0}^{\infty}\left\{\frac{q^{-1}-q}{q+q^{-1}}\right\}^n\frac{q^{n(3n-1)}}{[n]_{q2}!}a^{+2n}q^{nN}\otimes a^{2n}q^{-nN} \tag{11}$$

in terms of the boson realization (3) of $\mathrm{sl}_q(2)$.

Then, on certain quotient spaces $Q_\alpha^\pm(p)\otimes Q_\alpha^\pm(p)$ of $\mathcal{F}_q\otimes\mathcal{F}_q$ we can obtain four classes of R-matrices from four basic representations on $Q_\alpha^+(p)$ with even αp, $Q_\alpha^+(p)$ with odd αp, $Q_\alpha^-(p)$ with even αp and $Q_\alpha^-(p)$ with odd αp. For lack of space in this letter, we only write one of them on $Q_\alpha^+(p)$ with even αp as follows

$$\begin{aligned}(R)^{m_1^1m_2^1}_{m_1m_2}&=q^{(2(m_1+j)+\frac{1}{2})(2(m_2+j)+\frac{1}{2})}\delta^{m_1^1}_{m_1}\delta^{m_2^1}_{m_2}+\sum_{n=1}^{2j}\frac{(q^{-4}-1)^n}{[2]_q^{2n}[n]_{q2}!}\\&\quad\times q^{(2(m_1^1+j)+\frac{1}{2})(2(m_2^1+j)+\frac{1}{2})+2\Sigma_{i=1}^n(m_1-m_2+2i)}\prod_{l=1}^{n}[2(m_2+j-l+1)]_q\\&\quad\times[2(m_2+j-l)+1]_q\delta^{m_1^1}_{m_1+n}\delta^{m_2^1}_{m_2-n}\end{aligned} \tag{12}$$

It is worth pointing out that only when $q^p=1$ will the R-matrix given by (12) satisfy YBE, so we call it non-generic R-matrix. In the case of representation (8), (12) gives a completely new R-matrix

$$R = q^{5/4} \cdot \begin{bmatrix} q & & & & & & & & \\ & q & 0 & & & & & & \\ & 0 & q & & & & & & \\ & & & q & q^{-1}-q & 0 & & & \\ & & & 0 & q^{-1} & 0 & & & \\ & & & 0 & 0 & q & & & \\ & & & & & & 1 & q^{-1}-q & \\ & & & & & & 0 & 1 & \\ & & & & & & & & q^{-1} \end{bmatrix} \qquad \text{for } j=1. \qquad (13)$$

Of course, for some cases (12) also gives reduced R-matrices i.e. they can be obtained from standard R-matrices by letting $q^p = 1$. For example, through (12) the representation (9) gives a reduction of the standard R-matrix with spin 1.

Finally we point out that this letter is only a brief report of the systematic research on non-generic R-matrices for YBE. The following results are about to be published.

(1) The general structure of R-matrix for the indecomposable representation of $U_q(A_n)$

(2) The classification of non-generic R-matrices

(3) The Yang–Baxterization of non-generic R-matrices

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